PROPAGATION OF A SOLITARY WAVE IN A PLASMA ALONG A MAGNETIC FIELD

(RASPROSTRANENIE UEDINENNOI VOLNY V PLAZME VDOL' MAGNITNOGO POLIA)

PMM Vol.27, No.1, 1963, pp. 85-90

A.D. PATARAIA
(Tbilisi)

(Received February 20, 1962)

Riemann solutions in magnetohydrodynamics were first analyzed in [1-3]. There has as yet been no success in finding a similar solution for a two-component quasi-neutral plasma.

There are a number of investigations [4,5] of the nonlinear oscillations in a plasma across an external magnetic field, in which the dependence of all the quantities on the variable $\xi = z - vt$ is assumed, where v is the constant wave propagation velocity. Such investigations were carried out in [6-8] for waves being propagated in a plasma along a magnetic field $\mathbf{H}_0$. It was shown in [6,7] that a solitary compression wave moving at the velocity

$$\left[\frac{n_*(m_i + m_e) H_0^2}{8\pi (n_0 + n_*) n_0 m_i \cdot m_e}\right]^{1/2}$$

exists in a cold plasma.

Here m_e, m_i are the electron and ion mass, respectively; n_* the plasma density at the point $\xi = 0$ and n_0 at infinity ($\xi = \pm\infty$). Moreover, the following quasi-neutrality conditions were obtained [7] for a cold plasma: $v^2 \ll (m_e/m_i)c^2$.

Certain questions associated with the propagation of a solitary wave along a magnetic field in a two-component, quasi-neutral plasma in which binary collisions are lacking are examined below; the existence of magnetosonic solitary waves is investigated. In the case of a cold plasma it is found that the positive charge in a solitary compression wave is held around the maximum of the plasma density and the plasma is charged positively in this region but is negative at a distance from the maximum. For weak waves the value of the half-width and velocity of the waves as well as the quasi-neutrality condition for a hot plasma are obtained. Moreover, it is shown that the profile of a magnetosonic wave is deformed during its motion.

1. Fundamental equations. Let us assume that the wave moves along the z-axis along which the magnetic field $\mathbf{H}_0$ is directed. The respective motion equations for ions and electrons moving with velocity $\mathbf{v}_i$, $\mathbf{v}_e$ and with density n_i, n_e are

$$m_i n_i (v - u_i) \frac{d\mathbf{v}_i}{dz} = \text{grad } p_i - e n_i \left[E + \frac{\mathbf{v}_i \times \mathbf{H}}{c} \right] \tag{1.1}$$

$$m_e n_e (v - u_e) \frac{d\mathbf{v}_e}{dz} = \text{grad } p_e + e n_e \left[\mathbf{E} + \frac{\mathbf{v}_e \times \mathbf{H}}{c} \right] \tag{1.2}$$

Here u_i and u_e are the ion and electron velocities along the z-axis; p_i and p_e the partial pressures of the ion and electron gas, respectively; $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic field intensities, respectively; e the magnitude of the charge on the electron (ion); c the velocity of light.

Let us take the continuity equation in the form

$$\frac{d}{dz} [n_i (v - u_i)] = 0, \qquad \frac{d}{dz} [n_e (v - u_e)] = 0 \tag{1.3}$$

and the Maxwell equation as

$$\text{rot } \mathbf{H} = \frac{4\pi}{c} \mathbf{j}, \qquad \text{rot } \mathbf{E} = \frac{v}{c} \frac{d\mathbf{H}}{dz}, \qquad \text{div } \mathbf{H} = 0 \tag{1.4}$$

$$\text{div } \mathbf{E} = 4\pi e (n_i - n_e) \tag{1.5}$$

($\mathbf{j} = e (n_i \mathbf{v}_i - n_e \mathbf{v}_e)$ — current in the plasma)

The plasma is assumed quasi-neutral, i.e. $n_e = n_i = n$. We neglect the difference $(n_i - n_e)$ everywhere except in (1.5). This small difference specifies the appearance of the electrostatic field E_z, which is the reason for the motion of the heavy positive ions. Moreover, equation (1.5) permits an estimate of the limits of the validity of the assumption on the quasi-neutrality of the plasma.

Let us introduce the notation (1.6)

$$\psi = \frac{H_x + i H_y}{[8\pi n_0 (m_i + m_e) v^2]^{1/2}} \exp\left[\frac{i a v_{\parallel}}{2v} h\right], \quad a^2 = \frac{(m_i - m_e)^2}{m_i m_e}, \quad n' = \frac{n_0}{n}, \quad \eta = \frac{\xi}{l}$$

$$v_{\parallel}^2 = \frac{H_0^2}{4\pi n_0 (m_i + m_e)}, \qquad \frac{d\eta}{dh} = n', \qquad l = \left[\frac{m_i m_e c^2}{(m_i + m_e) 4\pi n_0 e^2}\right]^{1/2}$$

Here H_x and H_y are components of the magnetic field $\mathbf{H}$ along the x- and y-axes. The problem is steady-state in the ξ variables. Let us look for the solution for the solitary wave which is symmetric relative to $\xi = 0$. Hence, the boundary conditions will be: the velocities $\mathbf{v}_e$, $\mathbf{v}_i$, the field components E_x, E_y, E_z, H_x, H_y and their derivatives with respect to ξ tend to zero at $\xi = \pm\infty$. Moreover, the magnitudes of the density and pressure of the electron and ion gas tend to the undisturbed value (i.e. at $\xi = \pm\infty$).

Carrying out simple manipulations of (1.1), (1.4) and using (1.6), we can write the equalities

$$\frac{d^2\psi}{dh^2} = \left[n' - \frac{(a^2+4) v_{\parallel}^2}{4v^2}\right]\psi, \quad \psi^2 = (1 - n') + \frac{v_0^2}{v^2}\left[1 - (n')^{-\gamma}\right] \quad \left(v_0^2 = \frac{T_i + T_e}{m_i + m_e}\right) \tag{1.7}$$

$$E_z = \frac{(m_e - m_i)^2}{el}\left[\frac{\gamma (m_i T_i - m_e T_e)}{(m_i^2 - m_e^2) v^2}(n')^{-\gamma-1} - 1\right]\frac{dn'}{dh} \tag{1.8}$$

Here T_e and T_i are the undisturbed temperatures for the electron and ion gas, respectively; γ the adiabatic exponent, which is considered to be the same for the electron and ion gas ($\gamma = 1,\ 5/3,\ 2$).

According to (1.7), the function ψ can be selected as real. This latter property was used in obtaining the second equation of (1.7). The variable h in the system of waves corresponds to the time in Lagrangean coordinates. Then, ψ real means that the frequency of precession of the magnetic field $\mathbf{H}$ around $\mathbf{H}_0$ is independent of the plasma temperature. This latter is caused by the isotropy of the temperature.

It is assumed below that the plasma is equilibrium, i.e. $T_e = T_i$. Integrating (1.7), using the relation for ψ^2 mentioned in (1.7) here, we obtain

$$\frac{d\psi}{dh} = \pm\frac{(1 - (n')^2)^{1/2}}{v^2 2^{1/2}}\left\{\left[v^2 - v_+^2(n')\right]\left[v^2 - v_-^2(n')\right]\right\}^{1/2} \tag{1.9}$$

Here

$$v_\pm^2(n') = \frac{\gamma v_0^2}{(1 + n')}\left\{\left[I + (I_+(n') I_-(n'))^{1/2}\right] \pm \left[(I - I_+(n'))(I - I_-(n'))\right]^{1/2}\right\}$$

$$I = \frac{(a^2 + 4) v_{\parallel}^2}{4\gamma v_0^2} \tag{1.10}$$

$$I_{\pm}(n') = \frac{(1+n')[(n')^{-\gamma}-1]}{\gamma(1-n')} - \frac{[(n')^{-\gamma+1}-1]}{(\gamma-1)(1-n')} \pm$$
$$\pm\left\{\frac{(1+n')[(n')^{-\gamma}-1]}{\gamma(1-n')}\left[\frac{(1+n')((n')^{-\gamma}-1)}{\gamma(1-n')} - \frac{2((n')^{-\gamma+1}-1)}{(\gamma-1)(1-n')}\right\}\right]^{1/2} \tag{1.11}$$

It is easy to establish the following properties of the function $I_{\pm}(n')$:

$$I_{+}(1) = I_{-}(1) = 1 \quad \text{for } n' = 1, \qquad I_{+}(n') \geqslant 1,\ I_{-}(n') \leqslant 1 \quad \text{for } n' \neq 1$$

Here the equality sign for $I_{+}(n')$ corresponds to the case $\gamma = 2$, $n' > 1$ and for $I_{-}(n')$ to the case $\gamma = 2$, $n' < 1$.

2. The solitary wave. To obtain the solution in the form of a solitary wave, the sign before the square root in (1.9) is selected in such a manner that: if the specific sign before the radical corresponds to a negative value of h, then the opposite sign would correspond to the positive value of h. Since the solution is sought as continuous everywhere (up to the second derivatives inclusively), then the derivative must vanish at the point $h = 0$ (in this paper $n' \neq 0$, $n' \neq \infty$ throughout)

$$d\psi / dh = 0. \tag{2.1}$$

If the $\psi = 0$ case is excluded from the analysis, then the derivatives $d\psi/dh$ and dn'/dh vanish simultaneously. Condition (2.1) yields the following value for the velocity of the rapid $v_{+}^{2}(n_{*}')$ and slow $v_{-}(n_{*}')$ magnetosonic waves ($n_{*}' \neq 1$):

$$v_{\pm}^{2} = v_{\pm}(n_{*}') \tag{2.2}$$

Here n_{*}' is the extremal value of n'. The expression for $v_{\pm}$ takes on a particularly simple form for $\gamma = 2$, $I = 1$

$$v_{+}^{2} = v_{-}^{2} = \frac{(a^{2}+4)v_{\parallel}^{2}}{4n_{*}'} \tag{2.3}$$

In the limiting cases

$$v_{+}^{2} = \frac{(a^{2}+4)v_{\parallel}^{2}}{2(1+n_{*}')}, \qquad v_{-}^{2} = 0 \qquad \text{for } T_{e} = T_{i} = 0 \tag{2.4}$$

$$v_{+}^{2} = \frac{2\gamma v_{0}^{2}}{(\gamma-1)(1-n_{*}')}[(n_{*}')^{-\gamma+1}-1], \qquad v_{-}^{2} = 0 \qquad \text{for } v_{\parallel} = 0 \tag{2.5}$$

Expanding the expression for $v_{\pm}^{2}(n')$ in powers of $(1 - n')$ and retaining terms containing the coefficients 1, $(1 - n')$, $(1 - n')^{2}$, we obtain the results

For $I - 1 \gg |1 - n_*'|$

$$I \geqslant I_+(n_*') = 1 + \left[\frac{\gamma}{2} + \left(\frac{\gamma+1}{3}\right)^{1/2} \operatorname{sgn}(1-n_*')\right](1-n_*') + \frac{(\gamma+1)}{3!}\{(\gamma+1) + [3(\gamma+1)]^{1/2}\operatorname{sgn}(1-n_*')\}(1-n_*')^2 \quad (2.6)$$

we have

$$v_+^2 = \frac{(a^2+4)\,v_\parallel^2}{4}\left\{1 + \frac{1-n_*'}{2!} + \left[\frac{1}{4} - \frac{(\gamma+1)}{12(I-1)}\right](1-n_*')^2\right\} \quad (2.7)$$

$$v_-^2 = \gamma v_0^2\left\{1 + \frac{(\gamma+1)}{2}(1-n_*') + \frac{(\gamma+1)}{3!}\left[(\gamma+2) + \frac{1}{2(I-1)}\right](1-n_*')^2\right\} \quad (2.8)$$

For $(1 - I) \gg |1 - n_*'|$

$$I \leqslant I(n_*') = 1 + \left[\frac{\gamma}{2} - \left(\frac{\gamma+1}{3}\right)^{1/2} \operatorname{sgn}(1-n_*')\right](1-n_*') + \frac{(\gamma+1)}{3!}\{(\gamma+1) - [3(\gamma+1)]^{1/2}\operatorname{sgn}(1-n_*')\}(1-n_*')^2 \quad (2.9)$$

we have

$$v_+^2 = \gamma v_0^2\left\{1 + \frac{(\gamma+1)}{2}(1-n_*') + \frac{(\gamma+1)}{3!}\left[(\gamma+2) - \frac{1}{2(1-I)}\right](1-n_*')^2\right\} \quad (2.10)$$

$$v_-^2 = \frac{(a^2+4)\,v_\parallel^2}{4}\left\{1 + \frac{1-n_*'}{2} + \left[\frac{1}{4} + \frac{(\gamma+1)}{12(1-I)}\right](1-n_*')^2\right\} \quad (2.11)$$

For the existence of a solitary wave describing the character of the change in the plasma density and the magnetic field, determined by the quantity

$$\psi^2 \geqslant \frac{H_x^2 + H_\parallel^2}{8\pi n_0 (m_i + m_e)\, v^2}$$

simultaneous compliance is necessary with the following inequalities:

$$\psi^2 \geqslant 0, \qquad (d\psi/dh)^2 \geqslant 0, \qquad (dn'/dh)^2 \geqslant 0 \quad (2.12)$$

If the first two inequalities of (2.12) are satisfied for $dn'/dh = \infty$ at a point $h \neq 0$, then a solitary wave exists only for a change in the magnetic field. It follows from formulas (2.3) to (2.5), (2.7), (2.8), (2.10), (2.11) that a fast, solitary compression wave exists for $I \gg 1$; a fast solitary rarefaction wave exists for $I \ll 1$ only for the magnetic field; no solitary wave exists for $I = 1$.

The boundary conditions and formula (1.3) lead to the equality

$$u = u_i = u_e = (1 - n')\,v \tag{2.13}$$

We obtain from (2.4), (2.5), (2.7), (2.10) and (2.13)

$$\frac{d}{dn_*'}\,[u + v_-(n_*')] < 0 \tag{2.14}$$

The inequality (2.14) shows that the profile of a magnetosonic wave is deformed during its motion.

3. Distribution of particle charge and energy within a solitary wave. Under conditions of quasi-neutrality of the plasma, the electron and ion kinetic energy is determined, respectively, by the formulas (in the system of waves)

$$W_e = m_i\left\{\frac{(1-(n')^2)v^2}{2} - \frac{\gamma v_0^2}{(\gamma-1)}\,[(n')^{-\gamma+1} - 1]\right\} + \frac{m_e\,(n')^2}{2} \tag{3.1}$$

$$W_i = m_e\left\{\frac{(1-(n')^2)v^2}{2} - \frac{\gamma v_0^2}{(\gamma-1)}\,[(n')^{-\gamma+1} - 1]\right\} + \frac{m_i\,(n')^2}{2} \tag{3.2}$$

The ion kinetic energy increases as n' increases and the electron energy decreases. This proves that the ion is held in the neighborhood of a peak of the solitary wave. For a cold plasma ($T_e = T_i = 0$) in the domain $1 > n' > \sqrt{2}/2$, the ion kinetic energy is greater than the electron kinetic energy and, vice-versa in the domain $0 < n' < \sqrt{2}/2$.

It is assumed below that the plasma is cold. The second formula of (1.7) and (1.9) lead to the following form for the profile of the solitary compression wave:

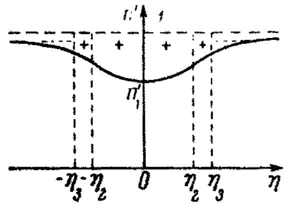
$$\left[\frac{2}{(1-n_*')}\right]^{1/2}\left\{\cosh^{-1}\left[\frac{1-n_*'}{1-n'}\right]^{1/2} - [(1-n_*')(n'-n_*')]^{1/2}\right\} = \pm\eta \tag{3.3}$$

where the plus and minus signs correspond to positive and negative η.

In order to establish the charge distribution within a solitary wave, let us write the expression for the plasma charge ($n_i - n_e$) by using (1.5), (1.7) to (1.9)

$$\frac{n_i - n_e}{n_0} = \frac{3\,(m_i^2 - m_e^2)\,v^2\,(1-n')}{m_i m_e c^2 n'}\left[\frac{(1+2n_*')}{3} - n'\right] \tag{3.4}$$

A qualitative picture of the profile and the charge distribution within a solitary wave is given on the figure. Points with the coordinates η_3 and $-\eta_3$ separate the positively from the negatively charged domains.



The value η_3 is determined by the equality

$$\eta_3 = \left(\frac{2}{1-n_*'}\right)^{1/2}\left[\cosh^{-1}\left(\frac{3}{2}\right)^{1/2} - \frac{(1-n_*')\,3^{1/2}}{2}\right] \quad (3.5)$$

The coordinates η_2 and $-\eta_2$ correspond to inflection points of the curve of the solitary wave. The inequality $\eta_2 < \eta_3$ is satisfied for all n_*'.

Integrating (3.4) in the domains $(0, \eta_3)$ and $(\eta_3, +\infty)$, we see that the obtained expressions are equal in magnitude but opposite in sign; this indicates the complete electrical neutrality of the plasma on the right branch of the solitary wave. The calculation for the left branch of the solitary wave is carried out analogously.

4. Width of the solitary wave. Formulas (1.7), (1.9) yield an expression for dn'/dh; expanding the latter in powers of $(1-n')$ and retaining terms with coefficients $(1-n')$, $(1-n')^2$, we find after integration of the obtained result

$$1 - n' = \frac{1}{a_1^2}\operatorname{sech}^2\left[\left(\frac{1}{2a_1}\right)^{1/2}\eta\right]$$
$$\left(a_1^2 = \frac{1}{2(1-M_\parallel^{-2})} - \frac{(\gamma+1)}{6(M_0^2-1)} > 1,\ M_\parallel^2 = \frac{(a^2+4)v^2}{v_\parallel^2},\ M_0^2 = \frac{v^2}{\gamma v_0^2}\right) \quad (4.1)$$

Hence, the half-width of weak solitary waves and the magnitude of the amplitude depend on both the magnetic field H_0 and on the temperature of the plasma.

5. Quasi-neutrality of the plasma. Using (1.5), (1.7) to (1.9) and retaining terms containing $(1-n')$ and $(1-n')^2$ in the expression for $(n_i - n_e)$ we obtain

$$\frac{(n_i - n_e)}{n_0} = -\frac{4v^2(m_i^2 - m_e^2)(1-E_1)(1-M_\parallel^2)(1-n')}{m_i m_e c^2}\Big[1 - \Big(\frac{2E_1(\gamma+1)}{(1-E_1)} +$$
$$+\frac{3a_1^2}{2} - \frac{5(\gamma+1)}{4(\mu_0^2-1)} - 1 \quad (5.1)$$
$$\left(E_1 = \frac{\gamma T_i}{(m_i + m_e)c^2}\right)$$

A necessary condition for quasi-neutrality is

$$|n_i - n_e| \ll n_0$$

This leads to the inequalities

$$v^2 \ll \frac{m_e}{m_i} c^2, \qquad T_i \ll m_e c^2 \tag{5.2}$$

BIBLIOGRAPHY

1. Kaplan, S.A. and Staniukovich, K.P., Reshenie uravnenii magnitogazodinamiki dlia odnomernogo dvizheniia (Solution of the magnetogasdynamic equations for one-dimensional motion). *Dokl. Akad. Nauk SSSR*, Vol. 95, No. 4, 1954.

2. Kulikovskii, A.G., O volnakh Rimana magnitnoi gidrodinamike (On Riemann waves in magnetohydrodynamics). *Dokl. Akad. Nauk SSSR*, Vol. 121, No. 6, 1958.

3. Akhiezer, A.I., Liubarskii, G.Ia. and Polovin, R.V., Prostye volny v magnitnoi gidrodinamike (Simple waves in magnetohydrodynamics). *ZTF* Vol. 29, No. 8, 1959.

4. Adlam, J.H. and Allen, J.E., The structure of strong collision free hydromagnetic waves. *Phil. Mag.*, Vol. 3, No. 29, 1958.

5. Hain, K., Lüst, R. and Schlüter, A., Hydromagnetic waves of fine amplitude in a plasma with isotropic and nonisotropic pressure perpendicular to a magnetic field. *Rev. Mod. Phys.*, Vol. 32, No. 4, 1960.

6. Saffman, P.G., Propagation of a solitary wave along a magnetic field in a cold collision free plasma. *J. Fluid Mech.* Vol. 11, No. 1, 1961.

7. Pataraia, A.D., Rasprostranenie nelineinykh kolebanii plazmy vdol' magnitnogo polia (Propagation of nonlinear plasma oscillations along a magnetic field), I. *ZTF* Vol. 32, No. 2, 1962.

8. Pataraia, A.D., Rasprostranenie nelineinykh kolebanii plazmy vdol' magnitnogo polia (Propagation of nonlinear plasma oscillations along a magnetic field), II. *ZTF* Vol. 32, No. 5, 1962.

Translated by M.D.F.